

How to reconstruct finite topological spaces given their quotient-spaces

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To my dear Tumim, in Memoriam

Abstract: We deal with the problem of reconstructing a finite topological space to within homeomorphism given (also to within homeomorphism) the quotient spaces obtained by identifying one point of the space with each one of the other points.

Key words: quotient-spaces, finite topologies, reconstruction procedures.

1 Introduction

Let T be a set, whose elements we are going to call points, let $x, y \in T$ and $T(x, y) = \{\{z\} | z \in T - \{x, y\}\} \cup \{\{x, y\}\}$. Consider the map $f : T \rightarrow T(x, y)$ (sometimes called the natural map) defined by setting $f(z) = \{x, y\}$ when $z \in \{x, y\}$ and $f(z) = \{z\}$ when $z \in T - \{x, y\}$. In colloquial language, we say that $T(x, y)$ was obtained from T by an identification of x and y .

If \mathcal{T} is a topology on T , the pair (T, \mathcal{T}) is said to be a topological space; and the topology obtained by identifying x and y , denoted $\mathcal{T}_{x,y}$, is defined as the topology on $T(x, y)$ for which a set $S \in \mathcal{T}_{x,y}$ is open if and only if $f^{-1}(S)$ is an open set in \mathcal{T} . We refer to this operation as a *topological identification* and $(T(x, y), \mathcal{T}_{x,y})$ is the topological *quotient-space* obtained by this identification.

We deal with the following

Problem A: Suppose that \mathcal{T} is a topology defined on the finite set $T = \{1, \dots, n\}$. Let $y \in T$ and suppose we are given topologies U_1, U_2, \dots, U_{n-1} on $n-1$ sets Q_1, Q_2, \dots, Q_{n-1} , each one with $n-1$ points, and for which we know that there is a bijection $G : \{1, 2, \dots, n-1\} \leftrightarrow T - y$ such that, for each i in the range $1 \leq i \leq n-1$, we have a homeomorphism $U_i \simeq \mathcal{T}_{g(i),y}$. Find \mathcal{T} to within homeomorphism, that is to say, reconstruct \mathcal{T} starting from the topologies U_i of its $n-1$ quotient-spaces.

In the statement of this problem, we assume that the $n-1$ given spaces are quotient-spaces of a topological space. By other words, we assume that there is a solution; our problem is to reconstruct it. As seen in detail in Section 8, if we are simply given $n-1$ topological spaces, each one of them with $n-1$ points, then, in general, there is no topological space which yields them by topological identification.

To clearly show the kind of work we are facing here, look at the following topology and its quotient-spaces, as they will be given:

Example 1: With $T = \{1, 2, 3, 4\}$ and $y = 4$, suppose we are given the open sets of topologies on three quotient-spaces: $U_1 \equiv \{a, b, c\}$ with $\emptyset, \{a\}, \{a, b, c\}$, $U_2 \equiv \{x, y, z\}$ with $\emptyset, \{x\}, \{x, y, z\}$, $U_3 \equiv \{g, h, j\}$ with $\emptyset, \{g\}, \{h\}, \{g, h\}, \{g, h, j\}$. A solution is \mathcal{T} with $\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}$ as open sets.

Note that non-homeomorphic topologies may yield the same quotient-spaces:

Example 2: Let $T = \{1, 2, 3, 4, 5\}$ and $y = 5$. Let \mathcal{T}_1 have $\emptyset, \{1, 2\}$ and T as open sets and \mathcal{T}_2 have $\emptyset, \{1, 2, 5\}$ and T as open sets. Both yield two quotient-spaces with $\emptyset, \{u, v\}$ and $\{u, v, x, z\}$ as open sets plus two other quotient-spaces with \emptyset and $\{u, v, x, z\}$ as open sets.

A related but much weaker version of this problem is dealt with in [9]. The reconstruction of finite structures when some substructures or some related structures are given is a topic which has deserved the attention of several authors. Concerning finite topologies, In [4], a topological n -space supposed to be connected, T_0 and T_5 , was reconstructed (to within homeomorphism), given (also to within homeomorphism) the subspaces induced on its $(n - 1)$ -subsets. In Graph Theory, the Ulam conjecture is the most famous case (see, for instance, [5]); but see also several papers like [1], [2] and [8].

For Topology, we cite classical texts like [3] or [6] or more recent ones like [7], [11] and [12].

As applications for this type of problem, I believe they may be found in the Biomedical Sciences. Think of fusions of DNA segments of a gene and how to recover the gene from such fused sequences.

2 Notation

Without loss of generality, we always suppose that $y = n$. If we are given a family F of $n - 1$ topological spaces (each one defined on a set with $n - 1$ points) for which the existence of a bijection, as defined in the statement of the *Problem* above, is known, then we denote them, as already said, by Q_1, \dots, Q_{n-1} with subscripts; a superscript, say Q^i , means that this space is known to be the quotient-space obtained from (T, \mathcal{T}) by topological identification of points n and i : hence, superscripts and subscripts have different meanings!

We need some specific terminology: with (T, \mathcal{T}) denoting a topological space, where $T = \{1, \dots, n\}$, an m -system is an open set P of \mathcal{T} such that $P - \{n\}$ has m points. An *upper* (resp., *lower*) m -system of \mathcal{T} is an m -system which does not contain (resp., contains) n . A k -system P of \mathcal{T} , with $k > 1$, is called an *old* k -system when each point p of P ($p \neq n$) is contained in some m -system where $m \leq k - 1$, and is called a *new* k -system when at least one

point p of P ($p \neq n$) is not contained in any m -system with $m \leq k - 1$.

We denote by α^* the smallest open set containing $\alpha \in T$ and we call α^* the covering set of α ; obviously, in a finite topological space, the covering sets completely determine the topology. For a given value of k , we say that $\alpha (\neq n)$ is an *old point* when α^* is an m -system for $m < k$; when α^* is an m -system for $m = k$, we say that α is a *new point*. In an *old k -system*, all points distinct from n are old. In a *new k -system*, at least one point distinct from n is new. Note that, for each k , there might be points which are neither old nor new, namely those points α such that α^* is an m -system for $m > k$.

It follows from the definitions that the existence of an open m -set in a quotient-space requires the existence in (T, \mathcal{T}) of an open set with either m points, all distinct from n (an upper m -system), or $m + 1$ points, one of them being n (a lower m -system). More precisely: take Q^i (where i is a superscript!) and let z denote the set $\{i, n\}$; when $i \notin A$, a subset A of Q^i is open in Q^i if and only if A is open in (T, \mathcal{T}) ; when $i \in A$, then we have $(A - \{i\}) \cup \{z\}$ open in Q^i if and only if $A \cup \{n\}$ is open in (T, \mathcal{T}) . To adopt a shorter notation, when A, B, \dots are subsets of the topological space (T, \mathcal{T}) , we write AB to mean the union of the disjoint sets A and B , and An to mean the union of the disjoint sets A and $\{n\}$. Moreover, we list the open sets of a topology by just writing their elements.

3 Outline for a solution

Our strategy to reconstruct the topological space (T, \mathcal{T}) is to obtain its open sets or, more precisely, its m -systems, for increasing values of m . This iterative procedure requires three tasks, which may be presented as follows:

Suppose we have all m -systems for $m \leq k - 1$. Let $X_{k-1} = \{1, \dots, x\}$ be the set of points distinct from n and contained in them (obviously, for $k = 1$ we have $X_0 = \emptyset$). Moreover, suppose that we can identify (we'll see how to do this in Section 7), for each value of k , the so-called *old spaces* which are those obtained by identification of n with each one of the points in X_{k-1} , the remaining spaces being called here *new spaces*.

The first essential task at the k -th iteration is to form all k -systems whose existence in T is implied by the already known m -systems, $m \leq k - 1$. Conditions under which this is possible are stated in Theorem 1. Relatively to the k -th iteration, these are the *old k -systems*. We know which ones among them enter in each quotient-space. In fact, with $A \subseteq X_{k-1}$, an open set A of T yields an open set in each new quotient-space, but an open set An of T yields no open set in any new quotient-space. Hence, for each quotient-space, we can immediately find the number of open k -sets containing at least one point not in X_{k-1} . These are the new open k -sets: as already said, each one of them, say V , is a covering set; it contains at least one point, say α , such that $V = \alpha^*$. The respective k -systems are the *new k -systems*. Obviously,

in each *new* k -system, there is at least one point $\alpha \neq n$, called a *new point*, which does not appear in any m -system for $m \leq k - 1$.

Although not necessary to reach our goal of rebuilding \mathcal{T} , we may be willing to associate to each point, or, more precisely, to the symbol \mathcal{P}_j of each point of each space Q , a subset (proper or not) of $T(x, y)$ which we call the set of possible numbers of \mathcal{P}_j in Q ; in fact, in the process of renaming the points of Q , the set of numbers of \mathcal{P}_j is the set of points of $T(x, y)$ whose image, through the respective natural map $\Psi : T \rightarrow Q$, may be \mathcal{P}_j . Here, the so-called *clans*, to be defined in Section 5, play an essential role.

The second essential task is to find the *new* k -systems. There are two types of new k -systems: the *clean* ones are covering sets of all its points (except possibly n if they contain it); the *mixed* ones contain *old points* (points whose covering set is an m -system for $m \leq k - 1$).

The third essential task is to obtain the right *configuration* (or configurations) and to form the sets constituting it (or them). For each value of k , the respective *configuration* is the collection of the new k -systems of \mathcal{T} . Configurations are presented in the next section. There are 13 altogether. After obtaining all m -systems, $m \leq k$, we form the set $X_k = X_{k-1} \cup Y_k$, where $Y_k = \{x + 1, \dots, w\}$ is the set of points whose covering set is a k -system, we list the quotient-spaces which may become Q^{x+1}, \dots, Q^w (that means, may become *old spaces*) and, sometimes, we may restrict the set of possible superscripts for each one of the previously considered quotient-spaces. Moreover, if we keep a register of possible numbers for each point of each quotient-space, we may now restrict these possible numbers as well. This completes the k -th iteration of the procedure.

Since T is finite, the procedure terminates as soon as all covering sets (and, consequently, all open sets) of T are formed, with just one exception, namely, the set $\{1, \dots, n - 1\}$. In fact, if $\{1, \dots, n - 1\}$ is the union of open k -sets with $k < n - 1$, then $\{1, \dots, n - 1\}$ is an open set of the reconstructed topology \mathcal{T} . Otherwise, it is optional to consider it open or not: note that the set of all elements of any quotient-space Q is always the image, through the respective natural map $f : T \rightarrow Q$, of the open set $\{1, \dots, n\} \equiv T$ of \mathcal{T} . Obviously, by the very definition of topology, nothing prevents us from taking also $\{1, \dots, n - 1\}$ as open set of \mathcal{T} . For an example, take two topologies on $T = \{1, 2, 3\}$, namely, \mathcal{T}' whose open sets are $\emptyset, \{1\}$ and $\{1, 2, 3\} \equiv T$, and \mathcal{T}'' whose open sets are $\emptyset, \{1\}, \{1, 2\}$ and $\{1, 2, 3\} \equiv T$: they yield the same quotient-spaces.

4 The 13 configurations

Let again A, B, \dots and R, S, \dots be subsets of the topological space (T, \mathcal{T}) .

Lemma 1 *If, at the k -th iteration, AR is a new open k -set where A contains only old points and R contains only new points, then A is open.*

Proof: Let $A = \{\alpha_1, \dots, \alpha_i\}$. For $j = 1, \dots, i$, we have $\alpha_j^* \subseteq AR$. Since no point of R is contained in an open set with less than k points, we may write $\alpha_j^* \subseteq A$. Hence, $\cup_j \alpha_j^* = A$.

Lemma 2 *If, at the k -th iteration, ARn and BSn are new open k -sets where A and B contain only old points whereas R and S contain only new points, then An and Bn are open.*

Proof: As in Lemma 1, for each α_j in A and each β_j in B , we have $\alpha_j^* \subseteq An$ and $\beta_j^* \subseteq Bn$. Moreover, noting that R and S are disjoint, we get $n^* \subseteq ARn \cap BSn = An \cap Bn$. Hence, $n^* \cup (\cup_j \alpha_j^*) = An$ and $n^* \cup (\cup_j \beta_j^*) = Bn$.

Lemma 3 *If, at the k -th iteration, ARn is a new open k -set, then at least one of the two sets A or An is open.*

Proof: No α_j^* can contain any point of R but it may contain n . Hence $\cup \alpha_j^*$ is A or An .

Lemma 4 *If, at the k -iteration, AR and ARn as well as BS and BSn are open k -sets, then A and An as well as B and Bn are also open sets.*

Proof: By Lemma 1, AR open implies A open and BS open implies B open. Moreover, $n^* \subseteq ARn \cap BSn = An \cap Bn$, hence $n^* \subseteq An$ and $n^* \subseteq Bn$ which implies $\cup \alpha_j^* \cup n^* = An$ and similarly, with an obvious notation, $\cup \beta_j^* \cup n^* = Bn$.

Theorem 1 *All k -systems whose points, distinct from n , are in X_{k-1} (that is, are old points at the k -th iteration) may be obtained as unions of k' -systems ($k' < k$), except possibly n^* (when it is also a k -system).*

Proof: First, let $A = \{\alpha_1, \dots, \alpha_k\}$ be an upper k -system and let $A \subseteq X_{k-1}$. This means that, for $i = 1, \dots, k$, α_i^* is a k' -system ($k' < k$) and $n \notin \alpha_i^*$. Hence, $A = \cup_i \alpha_i^*$, which proves the assertion in this case.

Now take a lower k -system, say $An = \{\alpha_1, \dots, \alpha_k, n\}$ with $A \subseteq X_{k-1}$, as in the preceding case. Again we have $An = \alpha_1^* \cup \dots \cup \alpha_k^* \cup n^*$ where each α_i^* is a k' -system ($k' < k$). Suppose that $n^* \neq An$. This means that $\{n\} \subseteq n^* \subset An$, that is to say, n^* is also a k' -system with $k' < k$. This completes the proof.

The reasoning in this proof fails when $n^* = An$. In fact, we may have, for instance, $1^* = \{1\}$, $2^* = \{2\}$ as 1-systems and $n^* = \{1, 2, n\}$. Here $X_1 = \{1, 2\}$, $A = \{1, 2\} \subseteq X_1$ and $n^* = An = \{1, 2, n\}$ is a 2-system but is not the union of 1-systems.

The importance of this theorem lies in the fact that, in accordance with it, unless $n^* = An$ with $A \subseteq X_{k-1}$ and $|A| = k$, the number of new open k -sets in each quotient-space may be obtained when a list of all k' -systems, $k' < k$, is already known. Note that an upper k -system $A = \{\alpha_1, \dots, \alpha_k\}$ yields an open k -set in all quotient-spaces *except* $Q^{\alpha_1}, \dots, Q^{\alpha_k}$ and a lower k -system

$An = \{\alpha_1, \dots, \alpha_k, n\}$ yields an open k -set in $Q^{\alpha_1}, \dots, Q^{\alpha_k}$. The number of new open k -sets in the new quotient-spaces may therefore be determined without taking into account whether or not $n^* = An$, with $A \subseteq X_{k-1}$; the number of new open k -sets in the old quotient-spaces may be affected by only one unit.

Theorem 2 *For given k , the number of new open k -sets in each new quotient-space is s , $s + 1$ or $s + 2$, where $s \geq 0$ is an integer which depends on k .*

Proof: First note that, if A and An are both k -systems (we then say they are paired), then, by the definitions, all quotient-spaces contain either A or $(A - \{i\}) \cup \{z\}$ as open k -set. Further, if A is an upper, non-paired k -system, then A appears as open k -set in all quotient-spaces except those which correspond to the points belonging to A and, finally, if An is a lower, non-paired k -system, then $(A - \{i\}) \cup \{z\}$ appears as open k -set in the quotient-spaces Q^i which correspond to the new points belonging to A .

The assertion now follows very easily:

Suppose there are $s + 1$ new upper k -systems. Each new point appears either in only one k -system or in two paired k -systems, otherwise it would appear in a k' -system with $k' < k$. Consider the quotient-space Q^i . If i is a new point which appears in an upper, non-paired k -system, then Q^i contains s new open k -sets. If i appears in two paired k -systems or if i does not appear in k -systems, then Q^i contains $s + 1$ new open k -sets. If i appears in a lower, non-paired k -system, then Q^i contains $s + 2$ new open k -sets. This completes the proof.

We denote by μ_1 , μ_2 and μ_3 the number of new quotient-spaces with s , $s + 1$ and $s + 2$ new open k -sets, respectively, and we distinguish four cases:

- Case 1: $\mu_1 \neq 0$, $\mu_2 = \mu_3 = 0$;
- Case 2: $\mu_1 \neq 0 \neq \mu_3$, $\mu_2 = 0$;
- Case 3: $\mu_1 \neq 0 \neq \mu_2$, $\mu_3 = 0$;
- Case 4: $\mu_1 \neq 0$, $\mu_2 \neq 0$, $\mu_3 \neq 0$.

Since we do not rule out $s = 0$ in any case, we see that $s = 0$ in Case 1 means that there are no new k -systems in any new quotient-space.

Each one of these 4 cases may be yielded by one of several *configurations*.

Theorems 3, 4 and 5 list such configurations; for clarity, we shall use a graphical layout in which upper and lower systems are written in an upper and lower position, respectively.

Theorem 3 *In Case 1, the new k -systems form one of the following configurations, where, for any i , A_i is a set of old points, R_i is a set of new points, and μ_1 is the total number of new points:*

1a) $s + 1$ upper k -systems, with each new point appearing in exactly one of the sets R_i , say

$$A_1 R_1 \ ; \ \dots \ ; \ A_{s+1} R_{s+1} \ ;$$

1b) s pairs of k -systems (here, μ_1 is the total number of new spaces, alias, all old and new spaces receive these s open sets with k points), say

$$A_1 R_1 ; \dots ; A_s R_s ; \\ A_1 R_1 n ; \dots ; A_s R_s n ;$$

1c) the empty configuration, that is, no new systems (here $s = 0$);

1d) a set of lower k -systems, with each new point appearing in exactly one of the sets R_i , say

$$A_1 R_1 n ; \dots ; A_x R_x n .$$

Note that, in this theorem, 1c and 1d require $s = 0$ and $s = 1$, respectively; 1c may be considered as 1b for $s = 0$; and in 1a, 1b and 1d, the sets A_i of old points are arbitrary, that is, they may be empty or non-disjoint, for instance. Obviously, for $k = 1$, all sets A_i are empty.

Proof: By an argument like the one used in the proof of Theorem 2, we see that, in the present hypothesis, there cannot exist a new upper k -system and a new lower k -system unless they are paired. Moreover, if two paired new k -systems exist, then all new k -systems must be paired. The possible configurations are therefore those indicated.

In 1a, $R_1 \cup \dots \cup R_{s+1}$ contains all new points, otherwise there would be quotient-spaces with $s + 1$ new k -sets, which is a contradiction. Similarly, in 1d, $R_1 \cup \dots \cup R_x$ contains all new points, otherwise there would be quotient-spaces with no new k -set. This completes the proof.

We deal with the configurations of Cases 2 and 4 together.

Theorem 4 *Using A, B to denote sets of old points, and R, S to denote sets of new points, the new k -systems in Cases 2 and 4 form one of the following configurations, the last one being only possible in Case 4:*

2a) or 4a) *With $|R_1| + \dots + |R_{s+1}| = \mu_1$, $|R| = \mu_3$ and μ_2 counting the number of remaining new quotient-spaces which is also the number of points which appear only in k'' -systems with $k'' > k$:*

$$A_1 R_1 ; \dots ; A_{s+1} R_{s+1} ; \\ B R n ;$$

2b) or 4b) *With $|R_1| + \dots + |R_{s+1}| = \mu_1$, $|S_1| + \dots + |S_x| = \mu_3$, μ_2 again as in 2a) or 4a), and $B_1 \cap \dots \cap B_x \neq \emptyset$ (See that $B_1 \cap \dots \cap B_x \neq \emptyset$, otherwise $n^* = \{n\}$ and no upper non-paired system could exist):*

$$A_1 R_1 ; \dots ; A_{s+1} R_{s+1} ; \\ B_1 S_1 n ; \dots ; B_x S_x n ;$$

4c) *With $|R_1| + \dots + |R_{i-1}| = \mu_1$, $|R_{s+2}| + \dots + |R_x| = \mu_3$, and $\mu_2 \geq |R_i| + \dots + |R_{s+1}| > 0$ (Here, $A_i \cap \dots \cap A_x \neq \emptyset$, same reason as in the preceding paragraph):*

$$A_1 R_1 ; \dots ; A_{i-1} R_{i-1} ; A_i R_i ; \dots ; A_{s+1} R_{s+1} \\ A_i R_i n ; \dots ; A_{s+1} R_{s+1} n ; A_{s+2} R_{s+2} n ; \dots ; A_x R_x n .$$

Proof: As a first remark, note that, for $k = 1$, all sets A and B are empty, hence configurations 2b, 4b and 4c never occur. Otherwise, as in the preceding theorem, we see that the new k -systems cannot be all upper, all lower or all paired. Moreover, we cannot have configurations with paired k -systems and lower, non-paired k -systems but without upper, non-paired k -systems. Similarly, we cannot have configurations with paired k -systems and upper, non-paired k -systems but without lower, non-paired k -systems. Hence, a configuration must have upper and lower k -systems. If it has also paired ones, then $\mu_2 \neq 0$. If it has no paired ones, then it may be $\mu_2 = 0$ or $\mu_2 \neq 0$; in fact, in this hypothesis, $\mu_2 \neq 0$ is the number of points which appear only in k'' -systems with $k'' > k$. This completes the proof.

Theorem 5 *Using again the same notation as in the preceding theorems, the new k -systems in Case 3 form one of the following configurations:*

3a) *With $|R_1| + \dots + |R_{s+1}| = \mu_1$ and $\mu_2 > 0$ counting the number of remaining new quotient-spaces, which is also the number of points which appear only in k'' -systems with $k'' > k$:*

$$A_1 R_1 ; \dots ; A_{s+1} R_{s+1} ;$$

3b) *With $|R_1| + \dots + |R_{i-1}| = \mu_1$, $\mu_2 \geq |R_i| + \dots + |R_{s+1}|$, $A_i \cap \dots \cap A_{s+1} \neq \emptyset$:*

$$A_1 R_1 ; \dots ; A_{i-1} R_{i-1} ; A_i R_i ; \dots ; A_{s+1} R_{s+1} \\ A_i R_i n ; \dots ; A_{s+1} R_{s+1} n ;$$

3c) *With $|R_1| + \dots + |R_x| = \mu_2$ and μ_1 counting the number of the remaining quotient-spaces among the new ones:*

$$A_{x+1} R_{x+1} ; \dots ; A_{x+s} R_{x+s} ; \\ A_1 R_1 n ; \dots ; A_x R_x n ; A_{x+1} R_{x+1} n ; \dots ; A_{x+s} R_{x+s} n ;$$

3d) *With $|R_1| + \dots + |R_x| = \mu_2$ and $\mu_1 > 0$ counting the number of points which appear only in k'' -systems with $k'' > k$ which is the same as the number of remaining quotient-spaces among the new ones:*

$$A_1 R_1 n ; \dots ; A_x R_x n .$$

Note that, in the statement of this theorem, 3d requires $s = 0$; 3a, for $\mu_2 = 0$, is 1a; 3d, for $\mu_1 = 0$, is 1d. Note also that, for $k = 1$, we cannot have, in 3b, more than one paired system.

Proof: In this case, we cannot have one lower, non-paired k -system together with one upper, non-paired k -system in the same configuration; in fact, if r_1 is a new point which appears in a lower, non-paired k -system, and r_2 is a new point which appears in an upper, non-paired k -system, then Q^{r_1} has two more k -systems than Q^{r_2} , which is impossible in Case 3. Hence, the possible configurations are those indicated and the theorem is proved.

5 Clans: definitions and their role

Clans and configurations are the basic tools for the reconstruction of \mathcal{T} , or, more precisely, for finding solutions to *Problem A*. To begin with, we define

the concepts of *clan of open sets* and *clan of covering sets*.

We resort to Graph Theory (see [5] or [10]) and we associate to the finite topology \mathcal{T} a digraph G whose vertices are the (non-empty) open sets of \mathcal{T} and whose arcs are defined as follows: there is an arc from β to α , written (β, α) , when β is properly contained in α and there is no γ such that $\beta \subset \gamma \subset \alpha$. There is an exception to this rule: A will not be connected to An .

A *clan of open sets* is a *connected sub-digraph* of G ; we don't say a *connected component* of G because we don't require maximality. The designation *clan* evokes our concept of a family. When there is a directed path from β to α , we say that α is a descendant of β and β an ancestor of α . When there is one single arc (β, α) , we say that α is in the generation following the generation of β . A vertex with no ancestor is called a *root*. Since T is a descendant of every open set, when T is the only descendant of the root the *clan* is said to be *trivial*; and when we look at T itself as a root (with no descendant), we call T a *pseudo-clan*. Given a clan, a *subclan* is formed by one of its sets and the respective descendants. Two clans are said to be *isomorphic* when their associated digraphs are isomorphic and corresponding vertices in both digraphs are open sets with the same number of points.

The *clans of covering sets* are defined like the *clans of open sets*. Here the role of the open sets is played by the covering sets. In both cases, note that a clan may have more than one root, or, which means the same, a set may belong to more than one clan, each one of them with a distinct root. For instance, with $T = \{1, \dots, 8\}$, set $1^* = \{1\}$, $2^* = \{2\}$, $3^* = 4^* = \{1, 2, 3, 4\}$, $5^* = \{1, 5\}$, $6^* = 7^* = \{2, 6, 7\}$ and $8^* = T$. We can as well recognize here two clans which are distinct but not disjoint: one is rooted at $\{1\}$, another is rooted at $\{2\}$.

The following observations should be kept in mind:

Observation 1: By the very definitions, each paired system yields the same open set in all spaces; each lower non-paired system yields open sets in the spaces whose superscripts will be the points (distinct from n) contained in the system.

Observation 2: No new mixed paired k -system $A; An$ may exist with A containing a subset B such that Bn is a lower non-paired k' -system with $k' < k$. This would imply $A \cap B = B$ to be an open set which would form with Bn a paired system, against the hypothesis.

Observation 3: ABn and ACn cannot exist as new k -systems except when An is a k' -system for $k' < k$.

Based on the very definitions and on these observations, we get further:

Observation 4: A clan whose root is a lower non-paired system contains only lower non-paired systems.

Observation 5: A clan whose root is in a paired system may contain paired and lower non-paired systems, but no upper non-paired system.

Observation 6: A clan whose root is an upper non-paired system may contain all kinds of systems. But, by the preceding observation, a subclan

rooted in a paired system cannot contain upper non-paired systems.

Observations 7 and 8 are trivial consequences, very useful to identify clans:

Observation 7: A clan rooted on an upper non-paired system appears entirely in Q^j if and only if j is not a point of any system of the clan.

Observation 8: A clan rooted on a lower non-paired system appears entirely in Q^j if and only if j is one point of the root.

As a very useful auxiliary step to rebuild \mathcal{T} , look at the given quotient-spaces and identify as many clans as possible. At this stage, we don't try to assign names to the symbols which may, and usually do vary from space to space. But we'll be able to distinguish clans of lower non-paired systems, clans of upper non-paired systems and clans of paired systems. Moreover, in a first step of this process, we look for clans with as many generations and as many members as possible. Keep all these clans at hand while looking at the configurations.

Do not forget that a point represented by a certain symbol in one space may be represented by a different symbol in another space. Looking at the first example below, we may have, to represent the same clan, in Q^1 , $p; pq; pqr$ and, in Q^2 , $d; df; df g$. What really matters is how the sets of the clan relate among them.

It is also important to notice that some apparently isomorphic clans may be of different types, that is, one lower, the other upper. Two disjoint upper isomorphic clans appear together in spaces Q^j where j does not belong to their sets; two disjoint lower isomorphic clans never appear together; when one is upper and the other is lower, they may appear together in Q^j where j is the root of the lower one. Paired clans or, more precisely, clans of paired systems, appear in all spaces.

Keep in mind that in a clan of lower systems, the systems with more points appear in more spaces; and in a clan of upper systems, the systems with more points appear in fewer spaces. To mention clans rooted at a paired system is not a very rigorous terminology: we have in mind clans with descendants which are paired systems or, exceptionally, lower non-paired ones.

6 The role of the configurations

Let us look now at the configurations. The considerations of the preceding sections are very useful when we proceed with the reconstruction. As suggested in Section 3, we take the given quotient-spaces and, for each k , using the k -systems they contain, we try to find the configurations which might have given rise to such systems. Since we look for \mathcal{T} up to homeomorphism, let us choose as names for the points in the 1-systems the first natural numbers $\{1\}$, $\{2\}$, ..., $\{p\}$; by other words, give to the points of T which constitute covering sets with just one point or one point and n , the first integer numbers. In the next steps, we keep assigning successive integers to the new points in

the k -systems for $k = 2, 3, \dots$. Obviously, if there are no 1-systems, we start with 2-systems or, even more generally, we start with the smallest systems.

Configurations of type 1. When all new spaces have the same number, say s , of new k -systems, the possible configurations are 1a, 1b, 1c or 1d. We look separately at cases **A** ($s = 0$), **B** ($s = 1$) and **C** ($s > 1$).

A: When $s = 0$, two configurations are possible: 1a (with $s + 1 = 1$ and hence R_1 containing all new points) or 1c. If, in 1a, we have only A_1R_1 , that is, $s + 1 = 1$, hence $s = 0$, then we get $\mu_1 = |R_1|$ new spaces with 0 new k -systems, which means that the new k -system does not appear in any new space. How can we distinguish now 1a from 1c? If $A_1R_1 \neq \{1, \dots, n - 1\}$, then for $j \notin A_1R_1$, the space Q^j contains A_1R_1 . Of course Q^j is an old space, otherwise the configuration was not 1a, because not all the new spaces would have the same number of new k -systems. If $A_1R_1 = \{1, \dots, n - 1\}$, then there is no need for further discussion, because such a new $(n - 1)$ -system always exists. By other words, for $k = n - 1$, configuration 1c never occurs. See *Example 5* in the last section.

B: When $s = 1$, we may have three configurations: 1a, 1b or 1d. In 1a, we have two upper k -systems A_1R_1 and A_2R_2 ; in 1b, the paired k -systems A_1R_1 and A_1R_1n ; in 1d, one lower k -system A_1R_1n .

To distinguish these three configurations see that, in 1a, $|R_1| + |R_2| = \mu_1$ (all new points are in $R_1 \cup R_2$, hence $|R_1|, |R_2| < \mu_1$) and $|A_1R_1| = |A_2R_2| = k$; 1d differs from 1a because, in 1d, all new points are in A_1R_1n , hence $|R_1| = \mu_1$. Configuration 1b can also be distinguished from 1d. In 1d, either $A_1R_1 = \{1, \dots, n - 1\}$ which may always be considered an open set in 1b as well as in 1d, or there exists, in 1d, an old point j such that $j \notin A_1R_1n$, hence Q^j does not contain A_1R_1 ; but in 1b, all spaces contain A_1R_1 . Concerning 1a *versus* 1b, take A_1R_1 and A_2R_2 in 1a and check whether $|R_1| \neq |R_2|$ or whether there is an old point j such that $j \notin A_1R_1 \cup A_2R_2$ which implies that Q^j contains both A_1R_1 and A_2R_2 ; these conditions guarantee that we have 1a and not 1b. Sometimes we can't decide for 1a or 1b and we may get two non-homeomorphic solutions as we saw in *Example 2*. See also *Example 6* in the last section.

C: When $s > 1$, we may have two configurations: 1a or 1b.

In 1b, all spaces (old and new) contain s new k -systems, but, obviously, not necessarily all new points. In 1a, $R_1 \cup \dots \cup R_{s+1}$ contains all new points. In fact, if x is a new point and $x \notin R_1 \cup \dots \cup R_{s+1}$, then Q^x contains the $s + 1$ new k -systems, which contradicts 1a. If there is some $x \notin A_1R_1 \cup \dots \cup A_{s+1}R_{s+1}$, then Q^x has $s + 1$ new k -systems and it contains $A_1R_1, \dots, A_{s+1}R_{s+1}$. Of course Q^x is an old space; indeed, if Q^x were a new space, then, as we pointed out in the preceding case, the configuration was not 1a, because not all the new spaces would have the same number of new k -systems.

If no point x exists such that $x \notin A_1R_1 \cup \dots \cup A_{s+1}R_{s+1}$, then check the list of clans: in 1a we may detect distinct clans in the new spaces; in 1b, the clans

are the same in all new spaces. As an example, compare the quotient-spaces of two topologies on $T = \{1, \dots, 6\}$, namely \mathcal{T}' with open sets $1; 2; 13; 14; 25; \emptyset; T$ (where we distinguish clans $1; 13; 14$ and $2; 25$) and \mathcal{T}'' with paired systems $1; 1n; 2; 2n; 13; 13n; 25; 25n; \emptyset; T$. See the configurations for $k = 2$. When nothing helps, as in *Example 6* given in the last section, the reconstruction may lead to two distinct non-homeomorphic topologies: one through configuration 1a, the other through configuration 1b.

For $k = 1$, some of these configurations yield an immediate solution. In configuration 1a, $\mu_1 = n - 1$ and the reconstructed topology has all singletons as open sets, except $\{n\}$; with $n^* = \{n, a_1, \dots, a_r\}$, the pair $\{a_1, \dots, a_r\}$, $\{n, a_1, \dots, a_r\}$ yields $\{a_1, \dots, a_r\}$ in all spaces; this set, when $\beta \notin \{a_1, \dots, a_r\}$, appears in Q^β as an old r -system, otherwise it appears as new (see Example 2). In configuration 1b, when we have $s \geq 2$, we get $n^* = R_1 n \cap \dots \cap R_s n = \{n\}$ and, when $s = n - 1$, we get $1^* = \{1\}$, ..., $s^* = \{s\}$ which means that the reconstructed topology \mathcal{T} is the discrete topology. Finally, in configuration 1d, the reconstructed topology has all 1-systems $\{1, n\}$, ..., $\{n - 1, n\}$ as covering sets and, consequently, $n^* = \{n\}$.

Configurations of type 2 or type 4. As a first remark recall that, for 1-systems, because the sets A_i are empty, configurations 2b, 4b, 4c, and, as already pointed out, configuration 3b with more than one paired system never occur. They would imply $n^* = \{n\}$, hence no upper, non-paired 1-systems could exist.

Now, for $k \geq 1$, let us count the number of new k -systems in each new quotient-space. We distinguish configurations 2 from configurations 4 by the simple fact that, in configurations 2, some new spaces have s , others have $s + 2$, but no one has $s + 1$ new k -systems, and in configurations 4, besides those with s and $s + 2$, there are also new spaces with $s + 1$ new k -systems.

The way to distinguish between configurations 2a and 2b is an immediate consequence of the definitions: in 2a, the spaces with $s + 2$ systems exhibit all the new points in their new systems; in 2b, no space with $s + 2$ systems exhibits all the new points in its new systems, the reason being that in each one of these spaces only one of the two or more lower systems of 2b will be present. Don't forget that the new points in each new space are those which appear only in open sets with k or more points and, of course, never in sets with $k' < k$ points.

As regards configurations 4, it is important to register the points which have appeared as elements of open sets with $k'' < k$ elements and count them; count as well those which now appear in new spaces as elements of open sets with k points; this allows us to know how many are the remaining points, that is, those which appear in open sets with $k' > k$ elements. After doing this, the distinction between 4a and 4b is similar to what we did for 2a and 2b.

Configurations of type 3. It remains to consider the case when μ_1 new spaces have s new k -systems and μ_2 have $s + 1$ new k -systems, that is to say,

configurations 3. We look separately at the cases **A** ($s = 0$) and **B** ($s \geq 1$).

A: When $s = 0$, configurations 3b and 3c cannot occur; in these configurations, all new spaces have new systems. We may have 3a with $A_1 R_1$ as only k -system, hence $\mu_1 = |R_1|$ spaces with no new open k -set and all other new spaces with one new open k -set; or we may have 3d with $\mu_2 = |R_1| + \dots + |R_x|$ new spaces with one new open k -set and the remaining μ_1 new spaces with no new open k -set.

To distinguish 3a from 3d (when $s = 0$), we may usually resort to the number of new points in the new spaces with one new k -system. If this number is not the same for all these spaces, choose 3d. If it is the same, more has to be done to distinguish 3d from 3a: we then resort to the values of μ_1 and μ_2 . However, when $\mu_1 = \mu_2$, it may be impossible to make the distinction. Look at the topologies we give below, in *Example 3*.

Recall the topologies \mathcal{T}_1 and \mathcal{T}_2 of *Example 2*. For both of them, the only clans are trivial ones. When there are non-trivial clans, they can force upon us a choice between 3a and 3d.

Example 3: On the same set $T = \{1, 2, 3, 4, 5\}$, look at the following topologies: \mathcal{T}_3 with \emptyset , T , $\{1, 2, 5\}$ and $\{1, 2, 3, 5\}$ as open sets; \mathcal{T}_4 with \emptyset , T , $\{1, 2\}$ and $\{1, 2, 3\}$ as open sets; \mathcal{T}_5 with \emptyset , T , $\{1, 2\}$, $\{1, 2, 3\}$ and $\{1, 2, 4\}$ as open sets. One of the clans is rooted at a lower 2-system and it forces us to choose configuration 3d; two other clans are rooted at an upper 2-system and they force us to choose configuration 3a.

This can be recognized when we look at the spaces as they are given to us.

For $k = 1$, the configuration is 1c for all topologies.

For $k = 2$, all these topologies have $\mu_1 = 2$ spaces with $s = 0$ new 2-systems and $\mu_2 = 2$ spaces with $s = 1$ new 2-system. To decide between 3a (with 12 as only 2-system) and 3d (with $12n \equiv 125$ as only 2-system), we have to look at the clans. In \mathcal{T}_3 we recognize a progressive clan; in \mathcal{T}_4 we recognize a regressive clan. In \mathcal{T}_5 the spaces with $\alpha\beta; \alpha\beta\gamma$ as open sets are not produced by a lower clan; in some space the three systems of such a clan would appear; hence we have a clan with three upper systems. By other words, in \mathcal{T}_5 we have two spaces both with $\alpha\beta; \alpha\beta\gamma$ as open sets meaning $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ as open sets in the reconstructed topology; in \mathcal{T}_3 we have two spaces with $\alpha\beta; \alpha\beta\gamma$ plus one space with $\lambda\mu\nu$ as open sets meaning $\{1, 2, 5\}$, $\{1, 2, 3, 5\}$ as open sets in the reconstructed topology; in \mathcal{T}_4 we have one space with $\alpha\beta; \alpha\beta\gamma$ plus one space with $\varphi\psi$ as open sets meaning $\{1, 2\}$, $\{1, 2, 3\}$ as open sets in the reconstructed topology. After having the 2-systems, the old spaces Q^1 and Q^2 will be: for \mathcal{T}_1 , \mathcal{T}_4 and \mathcal{T}_5 the spaces with no 2-system; for \mathcal{T}_2 and \mathcal{T}_3 , the spaces with one 2-system.

For $k = 3$, we need no further observation, but it may be curious to verify that in \mathcal{T}_1 and \mathcal{T}_2 no new 3-system shows up in the new spaces, hence we have configuration 1c; in \mathcal{T}_3 and \mathcal{T}_4 , we have $\mu_1 = 1$ new space with $s = 0$ new 3-systems and $\mu_2 = 1$ new space with $s + 1 = 1$ new 3-system, which means

again that, if we do not look at the clans, we would be able to choose 3a or 3d; finally, in \mathcal{T}_5 , we have $\mu_1 = 2$ spaces with $s = 1$ new 3-system, hence the configuration to be chosen is 1a.

For $k = 4$, \mathcal{T}_1 and \mathcal{T}_2 have two new spaces and $\alpha\beta\gamma\delta$ is the 4-system which does appear in every space: it corresponds to configuration 1b; in \mathcal{T}_3 and \mathcal{T}_4 we have one new space and configuration 1b, again; finally, in \mathcal{T}_5 there is no new space for $k = 4$ and $\{1, 2, 3, 4\}$ is an old 4-system.

B: When $s \geq 1$, configuration 3d cannot occur, but 3a, 3b and 3c are possible. Let us present a few features which help us choose the right configuration.

Suppose we know which spaces are new and that we recognize the new k -systems. Let p be the number of new points in a space with $s + 1$ new k -systems. If not all new spaces with $s + 1$ new systems have the same number of new points, then we are done: we choose configuration 3c.

Suppose now that all new spaces with $s + 1$ new systems have the same number p of new points. If $\mu_1 \neq p$, configuration 3a is excluded; we can have only 3b (where $\mu_1 < p$) or 3c. If $\mu_1 = p$ configuration 3b is excluded; we can have only 3a or 3c.

Let now p' be the number of new points in a space with s systems. If p' is not the same for the μ_1 spaces with s new systems, then the configuration cannot be 3c, it must be 3a or 3b. Hence we can say: when p' is not the same for all spaces with s systems, we choose configuration 3a if $\mu_1 = p$, and configuration 3b if $\mu_1 < p$.

At this point we see that sometimes we have to find ways to distinguish 3a from 3c or 3b from 3c.

To distinguish 3a from 3c, look at the spaces with s systems: for configuration 3a, we have C_2^s combinations of these systems (C_2^s denotes the binary combinatorial coefficient, and for $s = 1$ we have $C_2^1 = 0$) and they appear, together with the $s + 1$ systems, in the spaces with $s + 1$ systems; for configuration 3c, in each one of the spaces with s systems, we have more than the C_2^s combinations of these systems. In fact, in this latter case the union of one lower, non-paired system, say $A_1 R_1 n$ with the lower set of a paired system, say $A_{x+1} R_{x+1} n$, yields a set which appears in Q^β for $\beta \in R_{x+1}$.

Sometimes it is possible to exclude 3b because the covering set n^* forbids the upper non-paired systems (when $n^* = \{n\}$, for instance). Moreover, if we have more spaces, we can make a choice: if these extra spaces have $s + 1$ new k -systems, then choose 3b; if they have s new k -systems, then choose 3c.

To distinguish 3b from 3c may be impossible, which means that we have two ways to proceed. Take the following example:

Example 4: Take $T = \{1, 2, 3\}$ with $n = 3$ and quotient-spaces Q_1 and Q_2 with topologies U_1 whose open sets are \emptyset, a, b, ab and U_2 whose open sets are \emptyset, u, uv . If seen as configuration 3b (for $k = 1$) we reconstruct \mathcal{T}' with $\emptyset, T, \{1\}, \{2\}, \{2, 3\}$ as open sets; if seen as configuration 3c (for $k = 1$) we reconstruct \mathcal{T}'' with $\emptyset, T, \{1, 3\}, \{2\}, \{2, 3\}$ as open sets. Nonetheless, check

that \mathcal{T}' and \mathcal{T}'' are homeomorphic: associate to points 1, 2, 3 of \mathcal{T}' , points 2, 3, 1, respectively, of \mathcal{T}'' .

7 The choice of old spaces for each value of k

As we said in Section 3, we keep assuming that we can identify the *old spaces* for each value of k . In this section we explain how to do this. Recall that, for each value of k , some k -systems are entirely formed by old points, that is, points which appear in m -systems for $m \leq k - 1$; a new k -system contains at least one new point, that means a point which has not appeared in any m -system for $m \leq k - 1$. Such points are new for k , but become old, and the spaces obtained by identification of n with each one of them will become *old spaces*.

Suppose we have identified the spaces whose superscripts belong to the set $\{1, \dots, x\}$. What we have to do now follows from the definition of the configuration.

In configuration 3a, we have μ_1 spaces with s new k -systems and the remaining μ_2 spaces with $s + 1$ new k -systems. Here the spaces with s new k -systems become old spaces: they get as superscripts $\{x + 1, \dots, x + \mu_1\}$. (For the moment forget the question *which is which*. In fact, sometimes, the quotient-spaces $Q^{\alpha_1}, Q^{\alpha_2}, \dots, Q^{\alpha_r}$ are isomorphic, hence they cannot be distinguished. This happens, for instance, when $\alpha_1^* = \alpha_2^* = \dots = \alpha_r^*$).

As regards configuration 1a, recall that it is configuration 3a when $\mu_2 = 0$.

In configuration 3d, we have μ_2 spaces with $s + 1 = 1$ new k -systems and the remaining μ_1 spaces with $s = 0$ new k -systems. Here the spaces with 1 new k -system become old spaces: they will receive as superscripts $\{x + 1, \dots, x + \mu_2\}$.

As regards configuration 1d, recall that it is configuration 3d when $\mu_1 = 0$.

In configuration 3c, we have μ_1 spaces with s new k -systems and the remaining μ_2 spaces with $s + 1$. These μ_2 spaces become old. Among those with s new systems, some of them also become old: to choose them, look at the unions of the paired k -systems with each lower non-paired k -system. The spaces which contain such unions become old; those which do not contain them remain new.

In configuration 3b, we have also μ_1 spaces with s new k -systems and μ_2 spaces with $s + 1$. The μ_1 spaces become old. Among the μ_2 spaces we choose to become old those with the whole clans (or sub-clans) rooted at the new k -systems.

Configuration 1b has s paired k -systems and it is configuration 3b with $\mu_1 = 0$ or configuration 3c with $\mu_2 = 0$. We choose to become old spaces the $|R_1| + \dots + |R_s|$ spaces with the biggest clans rooted at their new k -systems.

Concerning configuration 1c, we may have no new k -system or one clean new k -system with as many points as new spaces. The clean new k -system appears as a new k -system in all old spaces. When it exists, all new spaces become old.

For configurations 2a, 2b, 4a and 4b, it is easy to identify the μ_1 spaces with s new k -systems and the μ_3 spaces with $s + 2$ new k -systems. All these spaces become *old*. And if, as we did before, x denotes the number of spaces we have already classified as *old* while checking k' -systems with $k' < k$, then we assign to the μ_1 new spaces with s new k -systems superscripts from the set $\{x + 1, \dots, x + \mu_1\}$ and to the μ_3 new spaces with $s + 2$ new k -systems, superscripts from the set $\{x + \mu_1 + 1, \dots, x + \mu_1 + \mu_3\}$.

For configuration 4c, sometimes it is difficult to identify, among the spaces with $s + 1$ new k -systems, those which should receive a superscript from the set $\{x + \mu_1 + \mu_3 + 1, \dots, x + \mu_1 + \mu_3 + \mu\}$ where $\mu = |R_i| + \dots + |R_{s+1}|$. These values are associated to each one of the μ points in $R_i \cup \dots \cup R_{s+1}$; remember that we may have $\mu_2 > \mu$. To find μ , we do as follows: using the notation of *Theorem 4*, remember that $A_i \cap \dots \cap A_x \neq \emptyset$, hence the sets $A_i R_i, \dots, A_x R_x$ belong to a clan which appears in an old space. Let q be the number of new points in the clan, that means, points which appear only in open sets with k points. Knowing q , we obtain $q - \mu_3$ as the number of new points (and also of new spaces) in $A_i R_i \cup \dots \cup A_{s+1} R_{s+1}$.

The identification of these μ spaces may be difficult. Take, for instance, \mathcal{T} with $T = \{1, \dots, 9\}$, $n^* = 9^* = 1n$, $8^* = T$ and the following configurations:

$$\begin{aligned} &1 \ ; \ 23 \ ; \ 14 \ ; \ 15 \\ &1n \ ; \ ; \ 14n \ ; \ 15n \ ; \ 16n \ ; \ 17n \ ; \ 12345678n \end{aligned}$$

For $k = 1$ we have configuration 1b. For $k = 2$ we have configuration 4c where spaces Q^4 , Q^5 and Q^8 cannot be distinguished. For $3 \leq k < 8$, we have configuration 1c. For $k = 8$, we have configuration 1d.

If we modify \mathcal{T} by choosing $8^* = 178$ as covering set of 8 (which, together with $17n$, implies $178n$ as open set as well), the distinction of the above spaces remains impossible because 178 appears in all spaces.

This distinction is possible, if, for instance, we have an upper, non-paired k -system for $k \geq 3$: with $T = \{1, \dots, 11\}$ and $8^* = \{8, 9, 10\}$, the set $\{8, 9, 10\}$, root of a trivial clan, does not appear in Q^8 , Q^9 or Q^{10} but it appears in all other spaces, making it possible to distinguish between Q^4 and Q^5 , on one side, and Q^8 , Q^9 and Q^{10} , on the other side. Another way to make this distinction is to look at the clan rooted at the pair $1; 1n$. If it would include $158n$ and $159n$, then we would recognize Q^5 as one of the μ spaces for $k = 2$.

If we have no way to recognize the μ spaces which become old among those with the $s + 1$ new k -systems, then we are free to choose them as we wish. In our example, if nothing distinguishes Q^4 , Q^5 and Q^8 , then we choose two of them to be Q^4 , Q^5 and to become old. The third one, namely Q^8 , is a new space for $k = 3$.

Just a remark concerning the question *which is which* we mentioned above. This question is irrelevant for the reconstruction. However, if we are interested, an analysis of the clans allows us to choose the superscript of each space among those previously assigned as possible. With the help of the clans, we may also assign the right number to the symbols which represent the points of each quotient space.

8 A reference to a more general problem

Recalling what we said in Section 1, let us point out the existence of a more general problem, to be stated as follows:

Problem B: Suppose we are given $n - 1$ topological spaces, each one of them with $n - 1$ points. Find a topological space (T, \mathcal{T}) with $T = \{1, \dots, n\}$ such that the given spaces are homeomorphic to those obtained by topological identification of n with each one of the other points of T .

If a solution exists, then we may find it following the methods we used to deal with *Problem A*. Thus, quite naturally, we may ask now for a necessary and sufficient condition for the existence of a solution.

We urge our readers to tackle this question in detail. Meanwhile, just look at the following preliminary facts:

1. Given a topological space (T, \mathcal{T}) , the $n - 1$ spaces formed by topological identification of n with each one of the other points yield, for $k = 1, \dots, n - 1$, one of the 13 configurations we have listed and the successive configurations never infringe the Observations 4 through 8 about clans.

2. Reciprocally, for *Problem B* to have at least one solution, the following is a minimal set of conditions that must be satisfied: For $k = 1, \dots, n - 1$, the new covering sets (or the unions of a new covering set with $\{n\}$) of the given (new) topological spaces always form one of the 13 configurations that we have listed in Section 4, and the successive configurations never infringe the Observations 4 through 8 about clans.

Two particular cases deserve a special reference. They can be considered extreme cases for the covering set n^* of point n . For them, it is an immediate consequence of the definitions that the following two statements are valid:

1. When there is a solution where $n^* = T$, only 1a, 1c and 3a may appear as configurations;
2. When there is a solution where $n^* = \{n\}$, only 1b, 1c, 1d, 3c and 3d may appear as configurations.

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